Fiber Bundles as Gauge Symmetry

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Why do we care?

- As we will soon see, **fiber bundles** are **versatile**. It gives you more control with fewer equations

- We can use the language of fiber bundles to recover **Maxwell's Equations**

- Maxwell's Equations govern the laws of electricity, magnetism, and light.

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

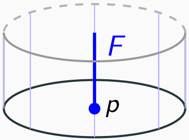
$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{E} = 0$$

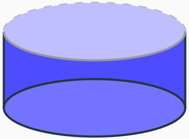
$$\nabla \cdot \mathbf{B} = 0$$

What is a Fiber Bundle?

- We will begin with a conceptual example from Tu. [Tu11]
- Our base space B, will be a **disk**.
- Consider a "**Fiber**" F, at a point p
- Then consider the "**Bundle**" of fibers to make the total space E, a cylinder.



Base space - B



Total Space - E

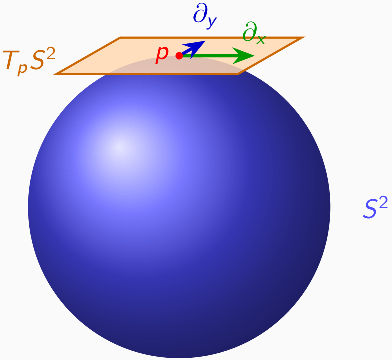
Vector Bundles

"It is no exaggeration to say that a clear understanding of just the one example of the tangent bundle suffices to give one a clear understanding of vector bundles in general."

— Sontz, *Principal Bundles* (2015), p. 31, [Son15]

The Tangent Space

- Let p be a point on the 2-sphere, S^2, and consider the tangent space at that point, T_pS^2.
- We can then consider every point $p \in S^2$, and take the disjoint union of all of the tangent spaces.
- Now we have the tangent bundle $TM = \bigcup_{p \in M} T_pM$ [Tu11]



Tangent Bundle Visualized

- Source: https://www.youtube.com/watch?v=lF_jSElv7ms

Fiber Bundles

A fiber bundle $(\boldsymbol{E}, \boldsymbol{\pi}, \boldsymbol{B}, \boldsymbol{F}, \boldsymbol{G})$ is defined with the following elements [Nak03]

- E - The Total Space - A differentiable manifold
- B - The Base Space - A differentiable manifold
- F - A Fiber - A differentiable manifold
- π - Projection - Surjective function
- G - Structure group - A Lie group

} Specifies how fibers are attached and glued

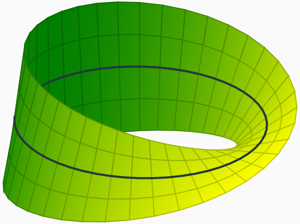
Shorthand notation: $E \xrightarrow{\pi} B$

Tangent Bundle as Fiber Bundle

- The Tangent Bundle works as a fiber bundle
- $B = S^2$ and $F = \mathbb{R}^2$ at every point p to represent each tangent space.
- $E = TS^2 = \{(p, v) : p \in S^2, v \in T_pS^2\}$
- The Hairy Ball Theorem - Link to Proof

Twists in a Fiber Bundle

- The **Möbius strip** is the simplest **nontrivial** fiber bundle.
    - $B = S^1, \quad F = (-1, 1) \subset \mathbb{R}$
    - Locally rectangular, $S^1 \times F$
    - But the fiber makes a half-twist as you go around, so globally $E \neq S^1 \times F$.
- Nontrivial: Bundle cannot be decomposed
- The cylinder from slide 1 *is* trivial.



Example of a Fiber Bundle - Hopf Fibration

- A **fantastic** object; highly deserving its own presentation
- Analogous to a spherical version of the 4-D hypercube
- `https://www.dynamicmath.xyz/hopf-fibration/app/`
- `https://youtu.be/AKotMPGFJYk?si=Ar7YXDXYPer6G23y`

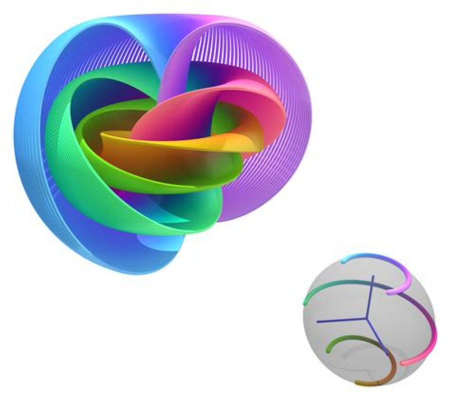


Figure 1: Source: Wikimedia Commons

Principal Bundles

- Transition functions t_{ij} are how we "*glue*" fibers from one patch to another. [Son15]
- These functions take values in the structure group G i.e. G controls the gluing.
- $G = F \implies$ "Principal Bundle."
- Example: **The Hopf Fibration**: $G = F = S^1$

Connections & Curvature

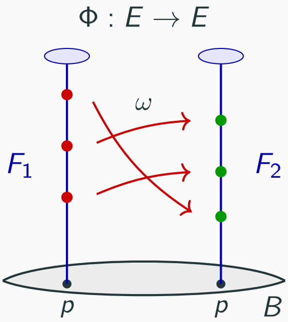
- We want to compare vectors in fibers.
- A **connection** is a 1-form ω on the principal bundle that tells us how to "parallel transport" along a path in B.
- **Curvature** is a measure of the failure of parallel transport.

Cartan's Structure Equation: $\boldsymbol{\Omega = d_P\omega + \omega \wedge \omega}$ [*Nak*03]

- If $\Omega = 0 \implies$ bundle is *flat* $\implies$ we are interested in $\Omega \neq 0$

Gauge Transformations as Bundle Automorphisms [Son15]

- A **gauge transformation** is a **bundle automorphism** $\Phi\colon E \to E$ that fixes every base point.
- We will reshuffle the fibers with the structure group, G
- Different shuffling describe the same physics - **gauge freedom**.



Physical Example: $U(1)$ Electromagnetism [Nak03]

$$U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\} \cong S^1$$

Bundle Language		**Physics Language**
Principal bundle $G = F = U(1)$	$\longleftrightarrow$	Gauge group $U(1)$
Base space $B = \mathbb{R}^{3,1}$	$\longleftrightarrow$	Minkowski spacetime
Connection 1-form ω	$\longleftrightarrow$	EM potential A_μ
Curvature 2-form $\Omega = d\omega$	$\longleftrightarrow$	Field tensor $F_{\mu\nu}$
Bundle automorphism	$\longleftrightarrow$	$A_\mu \mapsto A_\mu + \partial_\mu \lambda$

Electromagnetism is the geometry of a $U(1)$ principal bundle over spacetime.

Recovering Maxwell's Equations [Nak03]

Define the **gauge potential** 1-form and **field strength** 2-form:

$$\mathcal{A} = A_\mu \, dx^\mu, \qquad \mathcal{F} = d\mathcal{A} \,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Bianchi Identity

- $d\mathcal{F} = 0$
- Yields the pair:

Variation of the Maxwell Action

- $-\frac{1}{4} \int_{\mathbb{R}^4} F_{\mu\nu} F^{\mu\nu} d^4 x$
- Yields the pair:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0$$

All four Maxwell (vacuum) equations emerge from the geometry of the bundle.

References

[Nak03] Mikio Nakahara. **Geometry, Topology and Physics.** 2nd. Graduate Student Series in Physics. Taylor & Francis, 2003. ISBN: 978-0-7503-0606-5.

[Son15] Stephen Bruce Sontz. **Principal Bundles: The Classical Case.** Universitext. Springer, 2015. ISBN: 978-3-319-14764-2.

[Tu11] Loring W. Tu. **An Introduction to Manifolds.** 2nd ed. Universitext. New York: Springer, 2011. ISBN: 978-1-4419-7399-3.