The Klein-Gordon Equation on Anti-de Sitter Space

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Abstract

We study the Klein–Gordon equation for a scalar field on four-dimensional Anti-de Sitter spacetime (AdS_4). Starting from the vacuum Einstein field equations with a negative cosmological constant, we derive the global AdS_4 metric and compute the associated Laplace–Beltrami operator. Using the symmetries of the spacetime, we reduce the Klein–Gordon equation to an ordinary differential equation governing the radial behavior of the field. We then analyze the asymptotic behavior of solutions near the conformal boundary and show that they split into two distinct power-law branches. The corresponding exponents are determined by the field mass and lead to the Breitenlohner–Freedman bound, which constrains stability in AdS. We emphasize that, due to the timelike nature of the AdS boundary, boundary conditions are required to uniquely specify solutions, and different choices lead to physically distinct theories. Finally, we compare global AdS coordinates with the Poincaré patch to highlight the role of coordinate choices in understanding the spacetime's global structure.

1 Introduction and Motivation

We work in natural units $c = \hbar = 1$ with metric signature $(-+++)$.

Anti-de Sitter Space (*AdS* hereafter) is a maximally symmetric Lorentzian manifold with constant negative sectional curvature (saddle-like), whose metric is arises as a maximally symmetric solution to the Einstein Field Equations with negative cosmological constant.

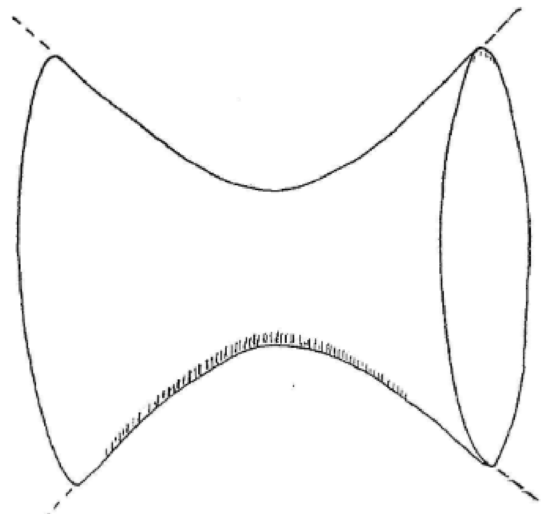

Figure 1: $1 + 1$ dimensional AdS spacetime. Fig. 2 of [3].

AdS is best known for its role in string theory and holography, specifically the AdS/CFT correspondence, which relates a theory with gravity in AdS to a theory without gravity defined on its boundary. The key idea of this correspondence is that what happens in the bulk is closely tied to behavior at the boundary. This makes the boundary conditions of our system of great importance. We will see how in the simple case of the Klein-Gordon equation, the choice of boundary conditions lead to distinct solutions and influences the behavior of the field throughout the spacetime.

The Klein-Gordon equation (KG equation hereafter) is a relativistic generalization of the Schrödinger equation for scalar fields. More specifically, it's a manifestly covariant differential equation which is second order in space and time, derived from the famous energy-momentum relation.

The global geometry of Anti-de Sitter spacetime has important properties. In particular, its conformal boundary is timelike. Unlike asymptotically flat spacetimes, signals can reach and return from this boundary in finite coordinate time, which makes the global structure physically relevant.

Studying the Klein-Gordon equation on AdS provides a concrete way to see how spacetime curvature affects relativistic fields. In particular, it highlights how boundary conditions at infinity play a direct role in determining the behavior of solutions.

2 Outline

In this project, we will explore the geometry of AdS_n with a fixed n. We will set $n = 4$ as is physically relevant in general relativity and derive the global metric for AdS_4 from Einstein's field equations for $\Lambda < 0$.

From that, we will compute the d'Alembertian (Laplace–Beltrami) operator for AdS_4 to analyze the KG equation. As a consistency check, we will apply to the Minkowski spacetime to note the differences between the two and verify our formula for $\Box$.

We will reduce the Klein–Gordon equation to an ordinary differential equation in the radial coordinate and analyze the asymptotic behavior near the conformal boundary.

Finally, we will investigate the Poincaré patch which is a coordinate chart on AdS where the metric becomes conformally flat. We will compare global and Poincaré coordinates and show that the Poincaré patch does not cover the entire spacetime.

3 Derivation of the Metric for AdS_4

To derive the AdS_4 metric, we begin with the vacuum Einstein Field Equations with cosmological constant [12].

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{1}$$

Here $T^{\mu\nu} = 0$, meaning we assume the spacetime contains no matter. Although unphysical on its own, this is not a serious limitation. Physical applications of AdS typically involve spacetimes that contain matter but approach pure AdS far from the matter sources, much like asymptotically flat spacetimes approach Minkowski space at large distances.

We will assume that our spacetime is maximally symmetric in 4 dimensions, we then have that the Riemann tensor is of the following form [6]

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right), \tag{2}$$

we obtain the Ricci tensor by contracting indices

$$R_{\sigma\nu} = g^{\rho\mu}R_{\rho\sigma\mu\nu}. \tag{3}$$

Substituting and simplifying,

$$R_{\sigma\nu} = \frac{R}{n(n-1)}\, g^{\rho\mu} \left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu} \right) \tag{4}$$

$$= \frac{R}{n(n-1)} \left(n\, g_{\sigma\nu} - g_{\sigma\nu} \right) = \frac{R}{n} g_{\sigma\nu}. \tag{5}$$

Where n is the spacetime dimension, which is 4 in our case. Thus,

$$R_{\mu\nu} = \frac{R}{4} g_{\mu\nu}, \tag{6}$$

Now we must obtain R in terms of Λ. To do this, we will contract the Einstein equations with $g^{\mu\nu}$

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R\, g^{\mu\nu} g_{\mu\nu} + \Lambda\, g^{\mu\nu} g_{\mu\nu} = 0. \tag{7}$$

Recall that

$$g^{\mu\nu} R_{\mu\nu} = R. \tag{8}$$

And additionally,

$$g^{\mu\nu} g_{\mu\nu} = \delta^{\mu}{}_{\mu} = 4 \tag{9}$$

Substituting these results back in, we obtain

$$R - \frac{1}{2} R \cdot 4 + \Lambda \cdot 4 = 0, \tag{10}$$

which simplifies to

$$R - 2R + 4\Lambda = 0 \quad \Rightarrow \quad R = 4\Lambda. \tag{11}$$

Substituting back into $R_{\mu\nu} = \frac{R}{4} g_{\mu\nu}$, we obtain our final form of the Einstein Field equations.

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{12}$$

To obtain the AdS_4 metric equation, we will assume that $\Lambda < 0$. We then ansatz a solution of the form

$$ds^2 = -f(r)^2\, dt^2 + f(r)^{-2}\, dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2 \right) \tag{13}$$

Substituting into the field equations gives a differential equation for $f(r)$ which forces $f(r)^2 = 1 + r^2/L^2$ [12], and we obtain the AdS global metric.

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2\, d\Omega^2$$

Where $d\Omega^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2$ and L is the AdS radius, defined by $L^2 = -\frac{3}{\Lambda}$

We call this the global metric because these coordinates cover the entire AdS spacetime: r ranges from 0 out to spatial infinity, and the angular coordinates parameterize a full sphere. This is in contrast to the Poincaré patch (Section 5), which only covers a portion of AdS.

If we let $r = L\sinh(\rho)$, we can see an alternative form of the metric which reinforces the notion that this is a hyperbolic spacetime.

$$ds^2 = -\cosh^2(\rho)\, dt^2 + L^2\, d\rho^2 + L^2 \sinh^2(\rho)\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right)$$

With the AdS_4 metric now established in two equivalent global coordinate systems, we turn to computing the d'Alembertian operator on this geometry in Section 4.

4 Laplace-Beltrami Operator

On flat 3D space, the Laplacian acts on a function f as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Extending to flat 4D spacetime, this becomes the d'Alembertian operator, $\Box$.

$$\Box f = -\partial_t^2 f + \nabla^2 f = \eta^{\mu\nu}\, \partial_\mu \partial_\nu f.$$

On a curved Lorentzian manifold, the natural generalization is the Laplace-Beltrami operator [6, 12], which we denote by Δ

$$\Delta f = \frac{1}{\sqrt{|g|}}\, \partial_\mu\left(\sqrt{|g|}\, g^{\mu\nu}\, \partial_\nu f\right).$$

This formula can be derived in two short steps. First, applied to a scalar, the covariant derivative reduces to the partial: $\nabla_\mu f = \partial_\mu f$, so $\Delta f = \nabla^\mu \nabla_\mu f = \nabla_\mu V^\mu$ for the vector field $V^\mu = g^{\mu\nu}\partial_\nu f$. Second, the covariant divergence of any vector field satisfies the identity [6]

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\, \partial_\mu\left(\sqrt{|g|}\, V^\mu\right),$$

which, applied to our V^μ, gives the Laplace-Beltrami formula.

Now let's apply this to AdS_4. Recall the metric from Section 3:

$$g_{\mu\nu} = \begin{bmatrix} -\left(1+\frac{r^2}{L^2}\right) & 0 & 0 & 0 \\ 0 & \left(1+\frac{r^2}{L^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}$$

with inverse

$$g^{\mu\nu} = \begin{bmatrix} -\left(1+\frac{r^2}{L^2}\right)^{-1} & 0 & 0 & 0 \\ 0 & 1+\frac{r^2}{L^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2(\theta)} \end{bmatrix}$$

Since the metric is diagonal, the determinant is the product of its diagonal entries:

$$g = \det(g_{\mu\nu}) = -\left(1+\frac{r^2}{L^2}\right)\cdot\left(1+\frac{r^2}{L^2}\right)^{-1}\cdot r^2\cdot r^2\sin^2(\theta) = -r^4\sin^2(\theta).$$

The $(1+r^2/L^2)$ factors cancel, leaving

$$|g| = r^4\sin^2(\theta), \qquad \sqrt{|g|} = r^2\sin(\theta).$$

To compute Δf, we substitute into the Laplace-Beltrami formula. Since the metric is diagonal, $g^{\mu\nu}$ contributes only when $\mu = \nu$, so the formula reduces to a sum of four independent terms. Pulling out the factors that don't depend on x^μ in each derivative:

$$\begin{aligned}
\mu = t: &\quad \frac{1}{r^2\sin(\theta)}\partial_t\left(r^2\sin(\theta)\frac{-1}{1+r^2/L^2}\partial_t f\right) = -\frac{1}{1+r^2/L^2}\partial_t^2 f \\
\mu = r: &\quad \frac{1}{r^2\sin(\theta)}\,\partial_r\left(r^2\sin(\theta)\left(1+\frac{r^2}{L^2}\right)\partial_r f\right) = \frac{1}{r^2}\,\partial_r\left(r^2\left(1+\frac{r^2}{L^2}\right)\partial_r f\right) \\
\mu = \theta: &\quad \frac{1}{r^2\sin(\theta)}\partial_\theta\left(r^2\sin(\theta)\frac{1}{r^2}\partial_\theta f\right) = \frac{1}{r^2\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta f\right) \\
\mu = \phi: &\quad \frac{1}{r^2\sin(\theta)}\partial_\phi\left(r^2\sin(\theta)\frac{1}{r^2\sin^2(\theta)}\partial_\phi f\right) = \frac{1}{r^2\sin^2(\theta)}\partial_\phi^2 f
\end{aligned}$$

The angular pieces (the θ and ϕ terms) combine into the familiar angular Laplacian, which we will denote Δ_{ang}.

$$\Delta_{\text{ang}} f = \frac{1}{\sin(\theta)}\,\partial_\theta(\sin(\theta)\,\partial_\theta f) + \frac{1}{\sin^2(\theta)}\,\partial_\phi^2 f.$$

Combining everything, the Laplace-Beltrami operator on AdS_4 reads

$$\Delta f = -\left(1+\frac{r^2}{L^2}\right)^{-1}\partial_t^2 f + \frac{1}{r^2}\,\partial_r\left(r^2\left(1+\frac{r^2}{L^2}\right)\partial_r f\right) + \frac{1}{r^2}\,\Delta_{\text{ang}} f.$$

With this expression for the Laplace-Beltrami operator on AdS_4, we are ready to analyze the Klein-Gordon equation in Section 6. But first, let's check for consistency.

4.1 Minkowski Limit

Let's see what happens as $L \to \infty$. The $(1 + r^2/L^2)$ factors go to 1, giving

$$\lim_{L\to\infty} \Delta f = -\partial_t^2 f + \frac{1}{r^2}\,\partial_r\big(r^2\,\partial_r f\big) + \frac{1}{r^2}\,\Delta_{\text{ang}} f.$$

The radial and angular pieces here are exactly the flat-space Laplacian ∇^2 written in spherical coordinates,

$$\nabla^2 f = \frac{1}{r^2}\,\partial_r\big(r^2\,\partial_r f\big) + \frac{1}{r^2}\,\Delta_{\text{ang}} f,$$

so the limit collapses to

$$\lim_{L\to\infty} \Delta f = -\partial_t^2 f + \nabla^2 f = \Box f,$$

recovering the flat-spacetime d'Alembertian as expected.

5 Poincaré patch

The defining property of a manifold is that it is locally Euclidean (more precisely, every point has a neighborhood homeomorphic to $\mathbb{R}^n$), and that it also has certain "nice" features that we would expect. [13, 11]

While AdS satisfies the conditions for a manifold, we can tell from a quick glance at ds^2 that it is certainly not as well-behaved as Minkowski space. To further analyze the space, we will consider the Poincaré patch.

The Poincaré patch will allow us to consider a section or "patch" of AdS. This will lead to a better understanding of the global structure of AdS, but we will also show that the Poincaré patch does not cover all of AdS and is thus not a global chart.

We define the Poincaré coordinates (t, x, y, z) where $z > 0$. These coordinates are related to the global coordinates by a change of variables[4, 10]. In these coordinates, the AdS_4 metric becomes [1]

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dx^2 + dy^2 + dz^2\right).$$

This form is significantly simplified compared to the global metric from Section 3. In fact, it is almost exactly the Minkowski metric, but with an overall factor of L^2/z^2.

[1] Note that Boschi and Braga use Λ for the AdS radius and not the cosmological constant as in this paper.

We say the metric is **conformally flat** because rescaling by z^2/L^2 produces flat spacetime. This is a major advantage since many calculations can be unwieldy in global coordinates, but they become tractable in the Poincaré patch.

Two limits stand out: $z \to 0$ and $z \to \infty$. As $z \to 0$, the conformal factor L^2/z^2 diverges, and we approach what is called the conformal boundary of AdS. The metric itself diverges at $z = 0$, but multiplying by z^2/L^2 recovers a finite Minkowski metric, which captures the asymptotic structure of AdS. This boundary is where the field theory in the AdS/CFT correspondence is defined [1, 9]. As $z \to \infty$, on the other hand, we hit a coordinate singularity called the Poincaré horizon.[10]

The existence of this horizon already tells us that the Poincaré patch does not cover all of AdS_4 [2, 4]. Geodesics that reach $z \to \infty$ in finite coordinate time leave the Poincaré chart, even though they continue regularly when viewed from global coordinates [2]. The Poincaré patch is therefore only a local chart. While it is useful for many calculations, it is insufficient for fully global statements about AdS [2].

With this comparison between global and Poincaré coordinates established, we now turn to the analysis of the Klein-Gordon equation in Section 6.

6 Analysis of the KG equation on the AdS boundary

We finally have all the pieces in place to study the Klein-Gordon equation on AdS_4. The KG equation for a scalar field ϕ of mass m on a curved spacetime is

$$(\Delta - m^2)\phi = 0,$$

where Δ is the Laplace-Beltrami operator we computed in Section 4. Substituting the explicit form gives

$$-\left(1+\frac{r^2}{L^2}\right)^{-1}\partial_t^2\phi + \frac{1}{r^2}\,\partial_r\left(r^2\left(1+\frac{r^2}{L^2}\right)\partial_r\phi\right) + \frac{1}{r^2}\,\Delta_{\text{ang}}\phi - m^2\phi = 0.$$

This is a partial differential equation in four variables: t, r, θ, and ϕ. Our strategy to solve it is to first reduce it to an ordinary differential equation in r alone using separation of variables, and then study what happens to solutions near the conformal boundary at $r \to \infty$.

6.1 Reduction to a Radial ODE

Separation of variables works here because the AdS_4 metric has two important symmetries. First, it is static (i.e. nothing in the metric depends on time). Second, it is spherically symmetric. The angular part of the metric is just $r^2 d\Omega^2$, the standard metric on a sphere. These symmetries let us look for solutions of the product form

$$\phi(t, r, \theta, \phi) = e^{-i\omega t}\, Y_{\ell m}(\theta, \phi)\, \chi(r),$$

where ω is a real frequency, $Y_{\ell m}$ are the standard spherical harmonics, and $\chi(r)$ is a radial function we want to determine.

We begin with the function depending on time. Differentiating $e^{-i\omega t}$ twice with respect to time gives

$$\partial_t^2 \, e^{-i\omega t} = (-i\omega)^2 \, e^{-i\omega t} = -\omega^2 \, e^{-i\omega t},$$

Thus,

$$\partial_t^2 \phi = -\omega^2 \phi$$

The angular piece relies on the fact that the spherical harmonics are eigenfunctions of the angular Laplacian.

$$\Delta_{\text{ang}} \, Y_{\ell m} = -\ell(\ell+1) \, Y_{\ell m}.$$

This is the same eigenvalue equation that governs the angular momentum operator $\hat{L}^2$ in the hydrogen atom, and the eigenvalues are the familiar $-\ell(\ell+1)$ from quantum mechanics.

Now we substitute $\phi = e^{-i\omega t} Y_{\ell m} \chi(r)$ into the KG equation and work through each term. The first term becomes

$$-\frac{1}{1 + r^2/L^2} \, \partial_t^2 \phi = -\frac{1}{1 + r^2/L^2} \, (-\omega^2) \phi = +\frac{\omega^2}{1 + r^2/L^2} \, \phi.$$

The third term, with the angular Laplacian, becomes

$$\frac{1}{r^2} \, \Delta_{\text{ang}} \phi = \frac{1}{r^2} \, (-\ell(\ell+1)) \phi = -\frac{\ell(\ell+1)}{r^2} \, \phi.$$

The radial term in the middle is unchanged in form, except that since $e^{-i\omega t}$ and $Y_{\ell m}$ don't depend on r, they pass through the radial derivative unchanged:

$$\partial_r \phi = e^{-i\omega t} Y_{\ell m} \, \frac{d\chi}{dr}.$$

Putting everything together and dividing through by the common factor $e^{-i\omega t} Y_{\ell m}$ (which is nonzero), the four-variable PDE collapses to an ODE for $\chi(r)$ alone:

$$\frac{1}{r^2} \, \frac{d}{dr} \left(r^2 \left(1 + \frac{r^2}{L^2} \right) \frac{d\chi}{dr} \right) + \frac{\omega^2}{1 + r^2/L^2} \, \chi - \frac{\ell(\ell+1)}{r^2} \, \chi - m^2 \chi = 0.$$

The dependence on t, θ, and ϕ has been completely absorbed into the constants ω and ℓ. We now have a single ODE we can study.

6.2 Asymptotic Behavior at the Conformal Boundary

We can't analytically solve this ODE for general ω, ℓ, m. But we can ask what does $\chi(r)$ look like for very large r, near the conformal boundary?

To answer this, let's see which terms in the ODE survive as $r \to \infty$ and which become negligible. We assume $r \gg L$. Going term by term we see that

- $1 + r^2/L^2 \approx r^2/L^2$, so $r^2\,(1 + r^2/L^2) \to r^4/L^2$.
- $\omega^2/(1 + r^2/L^2) \to \omega^2 L^2/r^2 \to 0$.
- $\ell(\ell+1)/r^2 \to 0$.
- m^2 is just a constant; it stays.

So the frequency and angular terms vanish at large r. Only the mass term and the leading part of the radial term survive. The asymptotic ODE is

$$\frac{1}{r^2}\,\frac{d}{dr}\left(\frac{r^4}{L^2}\,\frac{d\chi}{dr}\right) - m^2\,\chi = 0.$$

This is a simpler equation. Its structure suggests we look for power-law solutions, since taking a derivative of r^p just gives another power of r. So we ansatz a solution of the form

$$\chi(r) \sim r^{-\nu}$$

for some constant ν that we will determine later. We expect χ to decay rather than grow at large r, so ν is typically positive.

Now we just plug in and compute, step by step. Differentiating once gives

$$\frac{d\chi}{dr} = -\nu\, r^{-\nu-1}.$$

Multiplying by the prefactor r^4/L^2:

$$\frac{r^4}{L^2}\,\frac{d\chi}{dr} = \frac{r^4}{L^2}\cdot\left(-\nu\, r^{-\nu-1}\right) = -\frac{\nu}{L^2}\, r^{3-\nu}.$$

Differentiating again:

$$\frac{d}{dr}\left(-\frac{\nu}{L^2}\, r^{3-\nu}\right) = -\frac{\nu(3-\nu)}{L^2}\, r^{2-\nu}.$$

Dividing by the outer r^2:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^4}{L^2}\frac{d\chi}{dr}\right) = -\frac{\nu(3-\nu)}{L^2}\,r^{-\nu} = \frac{\nu(\nu-3)}{L^2}\,r^{-\nu},$$

where the last step is just rewriting $-\nu(3-\nu) = \nu(\nu-3)$. Substituting back into the asymptotic ODE:

$$\frac{\nu(\nu-3)}{L^2}\,r^{-\nu} - m^2 r^{-\nu} = 0.$$

Both terms have a factor of $r^{-\nu}$, which is nonzero, so we can divide it out:

$$\frac{\nu(\nu-3)}{L^2} - m^2 = 0,$$

or rearranging:

$$\nu(\nu-3) = m^2L^2.$$

This is just a quadratic equation in ν. Expanding:

$$\nu^2 - 3\nu - m^2L^2 = 0,$$

and the quadratic formula gives the two solutions

$$\nu_\pm = \frac{3 \pm \sqrt{9+4m^2L^2}}{2}.$$

So the equation has *two* power-law solutions at large r: one going as $r^{-\nu_-}$ and the other as $r^{-\nu_+}$. The general asymptotic form of $\chi(r)$ is then a linear combination of the two:

$$\chi(r) \sim A\,r^{-\nu_-} + B\,r^{-\nu_+}, \qquad r \to \infty,$$

where A and B are constants determined by the boundary conditions.

6.3 The Breitenlohner-Freedman Bound

The two values $\nu_\pm$ are real only if the expression under the square root is nonnegative:

$$9 + 4m^2L^2 \geq 0,$$

which rearranges to

$$m^2 \geq -\frac{9}{4L^2}.$$

This is the Breitenlohner-Freedman bound, often called just the BF bound [5].

Note that this bound allows negative values of m^2. In flat Minkowski space, $m^2 < 0$ corresponds to a tachyonic field, and the theory is unstable. On AdS, however, the negative spacetime curvature provides an effective restoring force that confines the field. This stabilizes scalars with slightly negative m^2, as long as the mass squared stays above the BF threshold $-9/(4L^2)$.

If m^2 falls below this threshold, the BF bound is violated as $\nu_\pm$ becomes complex. The asymptotic solutions then oscillate as $r \to \infty$ rather than decaying, and the field becomes unstable.

6.4 Boundary Conditions and the Final Picture

We found that the asymptotic solution

$$\chi(r) \sim A\, r^{-\nu_-} + B\, r^{-\nu_+}$$

contains two free constants A and B. To pin down a unique solution, we need to specify the relationship between them at the conformal boundary.

This is where the global structure of AdS becomes decisive. As we noted in the introduction, the conformal boundary of AdS is timelike signals can travel from any interior point to the boundary in finite coordinate time. This means AdS is not globally hyperbolic[8, 12]. Specifying initial data on a Cauchy surface (a constant-time slice) is not enough to determine the future of the field, because new information can flow in from the boundary at any later time.

To make the dynamics well-defined, we supply additional information at the boundary itself. Different choices of boundary conditions, for instance requiring $A = 0$ or $B = 0$, or some linear relation between them give genuinely different theories with different solutions [7].

7 Conclusion

On AdS, the Klein-Gordon equation does not have a unique solution given just initial data. The timelike conformal boundary makes the boundary conditions a physical input that we must specify, not a mathematical convenience that we can ignore.

We started from Einstein's equations with $\Lambda < 0$, we derived the AdS_4 metric, computed the Laplace-Beltrami operator on this geometry, reduced the Klein-Gordon equation to a single ODE in r, and found that solutions near the conformal boundary fall into two distinct branches indexed by $\nu_\pm$. The Breitenlohner-Freedman bound tells us when these branches are real, and the timelike conformal boundary tells us why we must choose between them.

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