

The Klein-Gordon Equation on Anti-de Sitter Space

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Introduction and Motivation

- We work in natural units $c = \hbar = 1$ with signature $(- + + +)$.
- Anti-de Sitter space (AdS) is maximally symmetric with constant negative curvature.
- Its conformal boundary is **timelike**, so boundary conditions matter.
- This makes AdS a clean setting for studying the Klein-Gordon equation.
- The geometry changes the field equation, and the boundary changes the solution.

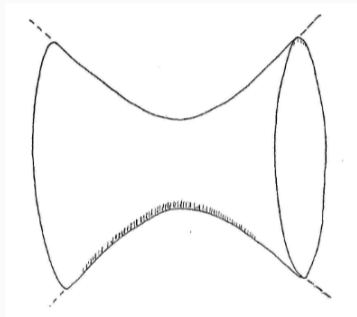
What is Anti-de Sitter Space?

- **Anti-de Sitter spacetime (AdS)**

- Vacuum solution to Einstein's equations with $\Lambda < 0$
- Maximally symmetric with constant negative curvature
- Hyperbolic geometry with a timelike conformal boundary

- **Why do we care?**

- Central spacetime in string theory and holography
- Foundation of the AdS/CFT correspondence
- Boundary conditions directly affect bulk physics



1+1 dimensional AdS

What is the Klein–Gordon Equation?

$$(\square - m^2)\phi = 0$$

- Relativistic wave equation for scalar fields
- Covariant generalization of the Schrödinger equation
- Comes from the energy/momentum relation

$$E^2 = p^2 + m^2$$

Link to KG visualization:

[https://youtu.be/RcyfarXgJaU?
si=KmH7Simunvmi0VNY&t=879](https://youtu.be/RcyfarXgJaU?si=KmH7Simunvmi0VNY&t=879)

Why do we care?

- Simplest field theory on curved spacetime.
- On AdS, boundary conditions become physically important

- Start from Einstein's equations with $\Lambda < 0$.
- Derive the global metric for AdS_4 .
- Compute the Laplace-Beltrami operator.
- Reduce the Klein-Gordon equation to a radial ODE.
- Study the large- r behavior and the BF bound.

Deriving the Metric

Start from the vacuum Einstein equations with cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

For a maximally symmetric 4D spacetime,

$$R_{\mu\nu} = \frac{R}{4}g_{\mu\nu}, \quad R = 4\Lambda,$$

so the equations reduce to

$$R_{\mu\nu} = \Lambda g_{\mu\nu}.$$

We take $\Lambda < 0$, which gives Anti-de Sitter space.

Global AdS_4 Metric

Use the static, spherically symmetric ansatz

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\Omega^2.$$

Solving the field equations gives

$$f(r)^2 = 1 + \frac{r^2}{L^2}, \quad L^2 = -\frac{3}{\Lambda}.$$

So the global metric is

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

Equivalent Global Coordinates

Let

$$r = L \sinh \rho.$$

Then the metric becomes

$$ds^2 = -\cosh^2(\rho) dt^2 + L^2 d\rho^2 + L^2 \sinh^2(\rho) d\Omega^2.$$

This makes the hyperbolic character of AdS easier to see.



A static slice of AdS
Escher tessellation

Laplace–Beltrami Operator

On a curved spacetime, the scalar Laplacian becomes

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu f \right).$$

For the AdS metric, the determinant is

$$|g| = r^4 \sin^2 \theta, \quad \sqrt{|g|} = r^2 \sin \theta.$$

The metric is diagonal, so the operator splits into time, radial, and angular pieces.

The final result is

$$\Delta f = - \left(1 + \frac{r^2}{L^2}\right)^{-1} \partial_t^2 f + \frac{1}{r^2} \partial_r \left(r^2 \left(1 + \frac{r^2}{L^2}\right) \partial_r f \right) + \frac{1}{r^2} \Delta_{\text{ang}} f.$$

where

$$\Delta_{\text{ang}} f = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta f) + \frac{1}{\sin^2 \theta} \partial_\phi^2 f.$$

As $L \rightarrow \infty$, this reduces to the flat-space d'Alembertian \square .

Klein–Gordon Equation on AdS

The curved-space Klein–Gordon equation is

$$(\Delta - m^2)\phi = 0.$$

Substituting the AdS Laplacian gives a PDE in t , r , θ , and ϕ :

$$-\left(1 + \frac{r^2}{L^2}\right)^{-1} \partial_t^2 \phi + \frac{1}{r^2} \partial_r \left(r^2 \left(1 + \frac{r^2}{L^2}\right) \partial_r \phi \right) + \frac{1}{r^2} \Delta_{\text{ang}} \phi - m^2 \phi = 0.$$

The symmetries of AdS let us reduce this to an ODE.

Separation of Variables

We separate the variables as such

$$\phi(t, r, \theta, \phi) = e^{-i\omega t} Y_{\ell m}(\theta, \phi) \chi(r).$$

The angular harmonics satisfy

$$\Delta_{\text{ang}} Y_{\ell m} = -\ell(\ell + 1) Y_{\ell m}.$$

This leaves a single radial function $\chi(r)$ to determine.

After substituting the ansatz and dividing out the angular and time dependence, we get

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \left(1 + \frac{r^2}{L^2} \right) \frac{d\chi}{dr} \right) + \frac{\omega^2}{1 + r^2/L^2} \chi - \frac{\ell(\ell + 1)}{r^2} \chi - m^2 \chi = 0.$$

This is the equation we analyze near the conformal boundary.

Asymptotic Behavior at the Boundary

For large r ,

$$1 + \frac{r^2}{L^2} \approx \frac{r^2}{L^2}, \quad \frac{\omega^2}{1 + r^2/L^2} \rightarrow 0, \quad \frac{\ell(\ell + 1)}{r^2} \rightarrow 0.$$

So the leading equation becomes

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^4}{L^2} \frac{d\chi}{dr} \right) - m^2 \chi = 0.$$

Trying $\chi(r) \sim r^{-\nu}$ gives

$$\nu(\nu - 3) = m^2 L^2.$$

Two Boundary Branches

The two exponents are

$$\nu_{\pm} = \frac{3 \pm \sqrt{9 + 4m^2L^2}}{2}.$$

So near the boundary,

$$\chi(r) \sim Ar^{-\nu_-} + Br^{-\nu_+}, \quad r \rightarrow \infty.$$

The constants A and B are fixed by boundary conditions.

The Breitenlohner-Freedman Bound

The exponents are real only when

$$9 + 4m^2L^2 \geq 0,$$

which gives

$$m^2 \geq -\frac{9}{4L^2}.$$

This is the Breitenlohner-Freedman bound.

It allows some negative m^2 , but not enough to make the field unstable.

Why the Boundary Matters

The AdS conformal boundary is timelike.

- Signals can reach the boundary and return in finite time.
- Initial data alone does not determine the full evolution.
- Boundary conditions are therefore part of the theory.
- Different choices of A and B give physically distinct solutions.

This is the main difference from the usual flat-space intuition.

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Conclusion

We started from Einstein's equations with $\Lambda < 0$ and obtained the global AdS_4 metric.

Then we computed the Laplace-Beltrami operator, reduced the Klein-Gordon equation to a radial ODE, and found two asymptotic branches near the boundary.

The BF bound determines when the field is stable, and the timelike boundary explains why boundary conditions are physically essential.